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LETTER TO THE EDITOR

Some solutions of the Zamolodchikov tetrahedron equation

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Abstract. By introducing a natural spectral parameter in the quantum Yang–Baxter equation, we construct a family of solutions of Zamolodchikov's tetrahedron equation. The general procedure is applied to the universal quantum group R-matrix.

Recently Carter and Saito [1] discovered a simple but quite remarkable relationship between the quantum Yang–Baxter (YB) [2] and the Zamolodchikov tetrahedron (ZT) [3] equations. In particular, this relationship allows one to construct solutions of the ZT equation from solutions of the YB equation. The procedure can be formulated in abstract algebraic terms as follows. Let $\mathscr{A}$ be an associative algebra (over C) with unity 1. *Consider any three elements* $\{A, M, B\}$ *of* $\mathscr{A}\otimes\mathscr{A}$, *satisfying the* YB

$$A_{12}A_{13}A_{23}=A_{23}A_{13}A_{12} \tag{1}$$

$$B_{12}B_{13}B_{23}=B_{23}B_{13}B_{12} \tag{2}$$

and the mixed equations

$$M_{12}M_{13}A_{23}=A_{23}M_{13}M_{12} \tag{3}$$

$$B_{12}M_{13}M_{23}=M_{23}M_{13}B_{12}. \tag{4}$$

Equations (1)–(4) represent in the standard notation equalities in $\mathscr{A}^{\otimes 3}$. In the following we call $\{A, M, B\}$ a Carter–Saito (CS) triplet. It is not difficult to show that any CS triplet gives rise to a solution of the ZT equation. One can proceed for instance by observing that A, M and B belong to $\mathscr{A}\otimes\mathscr{A}$ and therefore can be written in the form

$$A\equiv\sum_{i\in I} a_i\otimes a_i' \qquad B\equiv\sum_{j\in J} b_j\otimes b_j' \qquad M\equiv\sum_{k\in K} m_k\otimes m_k' \tag{5}$$

where for simplicity we assume that the index sets I, J and K are finite. More general *index sets can be treated along the same lines by requiring* $\mathscr{A}$ *to be a topological* algebra. Now, taking into account the conditions (1–4), one can verify by purely algebraic manipulations that

$$Z=\sum_{i\in I}\sum_{j\in J}\sum_{k\in K}[a_i\otimes m_k]\otimes[a_i'\otimes b_j]\otimes[m_k'\otimes b_j'] \tag{6}$$

satisfies the ZT equation

$$Z_{123}Z_{145}Z_{246}Z_{356}=Z_{356}Z_{246}Z_{145}Z_{123} \tag{7}$$

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on $[\mathscr{A}\otimes\mathscr{A}]^{\otimes 6}$. Clearly, in order to implement effectively the above method for deriving solutions of the ZT equation, one should solve the preliminary problem of constructing CS triplets. This is precisely the problem we address in the present note.

In what follows the YB equation with spectral parameter plays a fundamental role. For this reason we start by briefly recalling the approach to the spectral YB equation developed in [4]. Given a solution

$$R=\sum_{i\in I} c_i\otimes c_i'\in\mathscr{A}\otimes\mathscr{A} \tag{8}$$

of the YB equation, we have shown in [4] how to reconstruct a relative semigroup of spectral parameters $\mathscr{S}(R)$ belonging to End($\mathscr{A}$). More precisely, denoting by $\mathscr{A}_l$ and $\mathscr{A}_r$ the subalgebras of $\mathscr{A}$ generated by the elements $\{1, c_i: i\in I\}$ and $\{1, c_i': i\in I\}$, respectively, one has that $R\in\mathscr{A}_l\otimes\mathscr{A}_r$. Now we define the subset $\mathscr{S}_l(R)\subset\mathrm{End}(\mathscr{A}_l)$ as follows: $\alpha\in\mathscr{S}_l(R)$ if and only if there exists $\beta\in\mathrm{End}(\mathscr{A}_r)$ such that

$$[\alpha\otimes\mathrm{id}](R)=[\mathrm{id}\otimes\beta](R). \tag{9}$$

It is easily seen that $\mathscr{S}_l(R)$ is actually a semigroup with respect to the composition of endomorphisms. Let us introduce the mapping

$$R\colon \mathscr{S}_l(R)\to\mathscr{A}\otimes\mathscr{A}$$

defined by

$$R(\alpha)\equiv[\alpha\otimes\mathrm{id}](R). \tag{10}$$

Using the property (9), one can easily show that $R(\alpha)$ satisfies the spectral YB equation

$$R_{12}(\alpha_1)R_{13}(\alpha_1\alpha_2)R_{23}(\alpha_2)=R_{23}(\alpha_2)R_{13}(\alpha_1\alpha_2)R_{12}(\alpha_1). \tag{11}$$

The argument of the second factor of both sides of (11) is the composition $\alpha_1\alpha_2$ of the endomorphisms α_1 and α_2. The order is essential since in general $\mathscr{S}_l(R)$ is non-commutative. We have argued in [4] that $\mathscr{S}_l(R)$ represents a set of generalized spectral parameters and that the above algebraic procedure can be considered as a sort of 'Baxterization' [5].

The right counterpart $\mathscr{S}_r(R)$ of the semigroup $\mathscr{S}_l(R)$ is introduced analogously and one can verify that

$$R(\beta)\equiv[\mathrm{id}\otimes\beta](R)\qquad \beta\in\mathscr{S}_r(R) \tag{12}$$

satisfies (11) as well. In general one has $\mathscr{S}_l(R)\neq\mathscr{S}_r(R)$, which gives rise to a sort of asymmetry. Notice however that if R satisfies the YB equation, so does $\sigma(R)$, where σ is the exchange operator

$$\sigma\colon a\otimes b\mapsto b\otimes a.$$

Moreover, one easily verifies that $\mathscr{S}_l(\sigma(R))=\mathscr{S}_r(R)$ and $\mathscr{S}_r(\sigma(R))=\mathscr{S}_l(R)$. Without loss of generality, one can concentrate therefore on $\mathscr{S}_l(R)$.

The idempotent elements of $\mathscr{S}_l(R)$

$$\mathscr{I}_l(R)\equiv\{\varepsilon\in\mathscr{S}_l(R)\colon \varepsilon^2=\varepsilon\} \tag{13}$$

play a distinguished role in the above scheme. In fact, from (11) it follows that the mapping (10) satisfies the YB equation in any idempotent point of $\mathscr{S}_l(R)$. Furthermore, it is an immediate consequence of (11) that

$$\{R(\varepsilon_1), R(\varepsilon_2\alpha\varepsilon_1), R(\varepsilon_2)\} \tag{14}$$

is a SC triplet for any $\alpha \in \mathcal{S}_l(R)$ and $\varepsilon_1, \varepsilon_2 \in \mathcal{S}_l(R)$. In this way one obtains a whole family of CS triplets, naturally generated by a solution of the quantum YB equation.

In the rest of the present note we illustrate the above procedure for constructing CS triplets, using as a starting point the universal R-matrix associated with a quantum group. Denote by $\{\alpha_i: i=1, \ldots, r\}$ a basis of simple roots of a complex simple Lie algebra $\mathcal{G}$ of rank r. Let $\langle \cdot, \cdot \rangle$ be the invariant inner product on the root space and let a_{ij} be the associated Cartan matrix. The quantum group $\mathcal{U}_q(\mathcal{G})$ is the associative algebra generated by $\{1, X_i^\pm, H_i\}$, which satisfy the following commutation and generalized Serre relations

$$\begin{aligned} &[H_i, H_j] = 0 \\ &[X_i^+, X_j^-] = \delta_{ij}[H_i]_q \\ &[H_i, X_j^\pm] = \pm a_{ij} X_j^\pm \end{aligned} \tag{15}$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q_i} (X_i^\pm)^\nu X_j^\pm (X_i^\pm)^{1-a_{ij}-\nu} = 0 \qquad \forall i \neq j. \tag{16}$$

Here q is a complex parameter, $q_i \equiv q^{\langle \alpha_i, \alpha_i \rangle}$ and

$$\begin{aligned} &[X]_q \equiv \frac{q^X - q^{-X}}{q - q^{-1}} \\ &\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!} \qquad [n]_q! \equiv [n]_q[n-1]_q \ldots [1]_q. \end{aligned} \tag{17}$$

It is known [6] that $\mathcal{U}_q(\mathcal{G})$ has actually the structure of a quasi-triangular Hopf algebra; the corresponding universal R-matrix† $R \in \mathcal{U}_q(\mathcal{G}) \otimes \mathcal{U}_q(\mathcal{G})$ satisfies the YB equation and has the form

$$R = \sum_{\gamma \in Z_+^r} q^{[t_0 + 1/2(H_\gamma \otimes 1 - 1 \otimes H_\gamma)]} P_\gamma(u_i, v_i) \tag{18}$$

where $t_0 \in \mathcal{H} \otimes \mathcal{H}$ corresponds to the canonical scalar product in the Cartan subalgebra $\mathcal{H}$ and

$$H_\gamma = \sum_{i=1}^r \gamma_i H_i \qquad \gamma = (\gamma_1, \ldots, \gamma_r) \in Z_+^r$$

with $Z_+ = \{n \in Z: n \geqslant 0\}$. Finally, $\{P_\gamma\}$ are homogeneous polynomials (with q-dependent coefficients) in the variables $u_i = X_i^+ \otimes 1$ and $v_i = 1 \otimes X_i^-$ satisfying

$$\deg_{u_i} P_\gamma = \deg_{v_i} P_\gamma = \gamma_i. \tag{19}$$

It follows from the general form of P_γ that $R \in \mathcal{U}_q b_+ \otimes \mathcal{U}_q b_-$, where $\mathcal{U}_q b_+$ and $\mathcal{U}_q b_-$ are the subalgebras generated by $[1, H_i, X_i^+\}$ and $\{1, H_i, X_i^-\}$ respectively. The right-hand side of (18) can be written more explicitly [7], but we do not enter into details here because what we need below is only the property (19).

Let us consider now the space C^r and let us define the composition

$$\lambda\mu \equiv (\lambda_1 \mu_1, \ldots, \lambda_r \mu_r). \tag{20}$$

† To be more precise, one should use at this point a completion of $\mathcal{U}_q(\mathcal{G})$ with respect to a suitable topology and $\otimes$ should be understood in the topological sense.

Equipped with this operation, C^r becomes an abelian semigroup with 2^r idempotent elements forming the set $\mathcal{I}$. Any $\varepsilon \in \mathcal{I}$ is a vector in C^r whose coordinates take the values 1 or 0.

Our next step is to observe that for any $\lambda \in C^r$ the mapping

$$\alpha_\lambda(H_i) = H_i \qquad \alpha_\lambda(X_i^+) = \lambda_i X_i^+ \tag{21}$$

defines an endomorphism on $\mathcal{U}_q b_+$. It is obvious from the homogeneity property (19) that

$$[\alpha_\lambda \otimes \mathrm{id}](R) = [\mathrm{id} \otimes \beta_\lambda](R) \tag{22}$$

where β_λ is the following endomorphism on $\mathcal{U}_q b_-$:

$$\beta_\lambda(H_i) = H_i \qquad \beta_\lambda(X_i^-) = \lambda_i X_i^-. \tag{23}$$

Notice that both α_ε and β_ε are well-defined for any $\varepsilon \in \mathcal{I}$. From our general discussion it follows also that

$$R(\lambda) \equiv [\alpha_\lambda \otimes \mathrm{id}](R) = \sum_{\gamma \in Z_+^r} q^{[t_0 + 1/2(H_\gamma \otimes 1 - 1 \otimes H_\gamma)]} P_\gamma(\lambda_i u_i, v_i) \tag{24}$$

satisfies (11) with spectral parameter in the semigroup C^r. From (24) one gets for the elements of the CS triplet (14) the expressions

$$R(\varepsilon_1) = \sum_{\gamma \in Z_+^r(\varepsilon_1)} q^{[t_0 + 1/2(H_\gamma \otimes 1 - 1 \otimes H_\gamma)]} P_\gamma(u_i, v_i) \tag{25}$$

$$R(\varepsilon_2 \lambda \varepsilon_1) = \sum_{\gamma \in Z_+^r(\varepsilon_1 \varepsilon_2)} q^{[t_0 + 1/2(H_\gamma \otimes 1 - 1 \otimes H_\gamma)]} P_\gamma(\lambda_i u_i, v_i) \tag{26}$$

$$R(\varepsilon_2) = \sum_{\gamma \in Z_+^r(\varepsilon_2)} q^{[t_0 + 1/2(H_\gamma \otimes 1 - 1 \otimes H_\gamma)]} P_\gamma(u_i, v_i) \tag{27}$$

where $Z_+^r(\varepsilon) \equiv \{\gamma \in Z_+^r \colon \varepsilon\gamma = \gamma\}$. In any irreducible representation of $\mathcal{U}_q(\mathcal{G})$, the series in the right-hand side of equations (24)–(27) are actually finite sums. The resulting matrices can be given the form (5) and by means of (6) one derives a family of matrix solutions of the ZT equation.

In order to become more familiar with the general expressions (24)–(27), it is instructive to work out a concrete example. Let $\mathcal{G}$ be the Lie algebra A_r and let us consider the fundamental representation of $\mathcal{U}_q(A_r)$. Then (24) gives the $(r+1) \times (r+1)$-matrix

$$R(\lambda) = q \sum_{a=1}^{r+1} E_{aa} \otimes E_{aa} + \sum_{\substack{a,b=1 \\ a \neq b}}^{r+1} E_{aa} \otimes E_{bb} + (q - q^{-1}) \sum_{a,b=1}^{r+1} \eta_{ab}(\lambda) E_{ab} \otimes E_{ba} \tag{28}$$

where

$$\eta_{ab}(\lambda) = \begin{cases} \lambda_a \lambda_{a+1} \ldots \lambda_{b-1} & 1 \leqslant a < b \leqslant r+1 \\ 0, & 1 \leqslant b \leqslant a \leqslant r+1. \end{cases} \tag{29}$$

Observe that the matrix (28) has an inverse for any $\lambda \in C^r$. Substituting for λ in (29) two arbitrary elements $\varepsilon_1, \varepsilon_2 \in \mathcal{I}$ and $\varepsilon_2 \lambda \varepsilon_1 \in C^r$, one derives from (28) three (invertible) matrices which represent a cs triplet. Varying $\lambda \in C^r$ and ε_1, $\varepsilon_2 \in \mathcal{J}$ one generates actually a whole family of matrix cs triplets.

In conclusion, we would like to emphasize that the construction of cs triplets described above works on general algebraic level; the basic objects are an associative algebra $\mathcal{A}$ and a solution $R \in \mathcal{A} \otimes \mathcal{A}$ of the quantum YB equation. This fact allows one to derive universal (representation independent) cs triplets, which take matrix form after fixing a representation of $\mathcal{A}$. We believe that further investigations in this framework will shed new light on the relationship between the YB and ZT equations.

References

[1] Carter J S and Saito M 1992 On formulations and solutions of simplex equations *Preprint*

[2] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

[3] Zamolodchikov A B 1981 *Commun. Math. Phys.* **79** 489

[4] Liguori A and Mintchev M 1992 *Phys. Lett.* **B275** 371

[5] Jones V F R 1991 *Int. J. Mod. Phys.* **6** 2035

[6] Drinfeld V G 1987 Quantum Groups *Proceedings of the International Congress of Mathematicians, Berkeley 1986* ed A M Gleason (Providence: AMS) pp 798–820

[7] Rosso M 1989 *Commun. Math. Phys.* **124** 307
Levendorskii S Z and Soibelman Y B 1990 *J. Geom. Phys.* **7** 241
Kirillov A N and Reshetikhin N 1990 *Commun. Math. Phys.* **134** 421